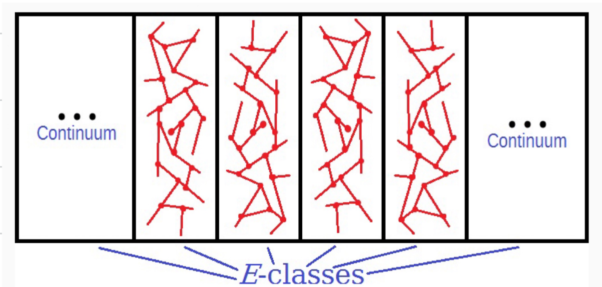
Ergodic Theory and Measured Group Theory
Lecture 23

A picture of a graphing of a CBER E:



Examples. (a) For any CBER E on X, E itself is basically a graphing, except that we chose graphs to be irreflexive, so $E \setminus \mathrm{Id}_X$ is indeed a graphing of E, called the complete graphing of E.

(b) Let $\Gamma \curvearrowright X$ be a Borel action of a ctbl group Γ and let $S \subseteq \Gamma$ be a symmetric generating set for Γ.
Recall that the Cayley graph of Γ wrt S is a graph $\mathrm{Cay}_S(\Gamma)$ on Γ with edges $(v, \sigma \cdot v)$ for all $v \in V$ and $\sigma \in S$.
Now for the action of Γ, we define its Schreier graph wrt S, as $G_S = \{(x, \sigma x) : x \in X, \sigma \in S\} \setminus \mathrm{Id}_X$. G_S is Borel because $(x,y) \in G_S \iff \underbrace{\exists \sigma \in S}_{\text{ctbl union}} \; \underbrace{y = \sigma x}_{\text{Borel of the diagonal}}$.

It's obvious that the orbit eq. rel. $E_\Gamma = E_{G_S}$, i.e. G_S is a graphing of E_Γ.

Obs. Each component of G_S is a homomorphic image of $\mathrm{Cay}_S(\Gamma)$, given by $\gamma \mapsto \gamma \cdot x$.
If the action is free, this is a graph isomorphism.

Prop. For any CBER E, all of its graphings are Schreier graphs.

Proof. Let G be a graphing of E.
By the Feldman–Moore theorem, $E = \bigcup_n \operatorname{graph}(\gamma_n)$ for some Borel involutions. Let $\gamma_n' : X \to X$ by $x \mapsto \gamma_n x$ if $(x, \gamma_n x) \in G$ and $x \mapsto x$ otherwise. Let $\Gamma = \langle \gamma_n' : n \in \mathbb{N} \rangle$. It's easy to see, $\Gamma \curvearrowright X$ and $E_\Gamma = E$. Letting $S := \{\gamma_n' : n \in \mathbb{N}\}$, we see that the Schreier graph of $\Gamma \curvearrowright X$ w.r.t. S is precisely G. $\square$

Now we define the analog of rank called cost. Again recall that the rank of a group Γ is minimum over all its Cayley graphs of $\frac{1}{2}$ the degree of each vertex. Also recall, that for a finite graph G, # edges of $G = \frac{1}{2} \sum_{v \in V(G)} \deg_G(x)$.

Equipping $V(G)$ with the counting measure, we can write this as $|E(G)| = \frac{1}{2} \int_{V(G)} \deg_G(x)\, d\mu(x)$.

Def. For any loc. ctbl Borel graph G on a prob. space (X,μ), its **cost** is: $C_\mu(G) := \frac{1}{2} \int_X \deg_G(x)\, d\mu(x) = \frac{1}{2} \int_X |G_x|\, d\mu(x)$

For a CBER E on (X,μ), its **cost** is

$$c_\mu(E) := \inf_G C_\mu(G),$$

where G ranges over all Borel graphings of E.

This notion of cost is only natural (by analogy with Cayley graphs) and useful for "pmp" CBERs.

Def. A CBER on (X,μ) is called **pmp** if $\forall$ Borel bijection $\varphi: A \to B$, for some Borel $A,B \subseteq X$, s.t. $\mathrm{graph}(\varphi) \subseteq E$ (i.e. $\varphi(x) E x$) we have $\mu(A) = \mu(B) = \mu(\varphi(A))$.

Prop. For a CBER E on (X,μ), TFAE:

(1) E is pmp

(2) $\forall$ Borel action $\Gamma \curvearrowright X$ of a ctbl gp Γ s.t. $E_\Gamma = E$, this

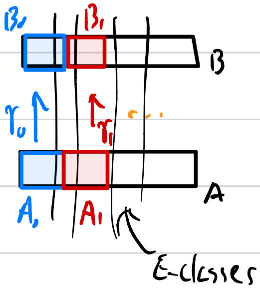
action is pmp.

(3) $\exists$ Borel action $\Gamma \curvearrowright X$ s.t. $E_\Gamma = E$ and this action is pmp.

Proof. (1) $\Rightarrow$ (2): Trivial because elements of Γ are Borel bijections with graphs $\subseteq E$.

(2) $\Rightarrow$ (3). By Feldman–Moore.

(3) $\Rightarrow$ (1). Let $A, B \subseteq X$ be Borel and $\varphi : A \to B$ be a Borel bijection. We implement φ as a piecewise translation by Γ with ctbly-many pieces: Fix $\Gamma = (\gamma_n)_{n\in\mathbb{N}}$ and $A_n := \{x \in A : n \text{ is the min. with } \gamma_n x = \varphi(x)\}$.



Then $\mu(B) = \mu\left(\bigsqcup_{n\in\mathbb{N}} \gamma_n A_n\right) = \sum_{n\in\mathbb{N}} \mu(\gamma_n A_n) = \sum_{n\in\mathbb{N}} \mu(A_n) = \mu(A)$. $\square$

We think of pmp as all points in the same E-class have equal mass, just like for finite groups. If a group Γ is finite then $\frac{1}{2}$ degree of each vertex in a Cayley graph $= \frac{1}{2}$ of the average degree (with the uniform prob measure on Γ).

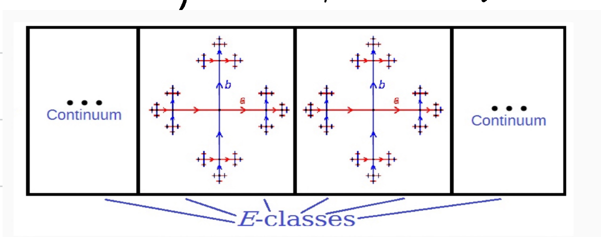
How do we compute the cost of E? What if we "take" a minimal graphing (i.e. acyclic). Would that achieve the cost? Wait, does minimal graphing (recall Borel) always exist?

Turns out that only special CBERs admit minimal graphings.

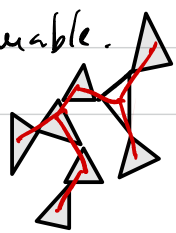
Def. A CBER on a st. Borel sp. X is called *treeable* if it admits an acyclic Borel graphing, called a *treeing*. If μ is a prob. meas. on X, we call E *μ-treeable* if E is treeable on a conull set (E-invariant if E is null-pres.) Groups all of whose free pmp actions induce treeable eq. rel. are called *strongly treeable*, and those that admit one such action are called *treeable*.

Open problem. Is treeable the same as strongly treeable.

Examples. (a) Free groups are strongly treeable, indeed the standard Schreier graphs of $\mathbb{F}_n$, $n \in \omega$, are acyclic.



(b) $\left(\mathbb{Z}/_{k\mathbb{Z}}\right)^{*}$ are strongly treeable.
E.g. $k=3$



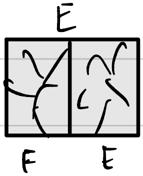
(c) Virtually free groups are st. treeable, e.g. $GL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})$.
(Jackson – Kechris – Louveau, uses geometric gp theory)

(d) Surface groups (i.e. fundamental gps of surfaces), and even elementary free groups (i.e. groups whose first-order theory = that of free groups).
(Conley – Gaboriau – Marks – Tucker-Drob)

(e) Hjorth showed there are continuum many Borel incomparable (wrt Borel reducibility) treeable eq. rel.

Nonexamples. (a) Kazhdan groups are anti-treeable, i.e. no free pmp action is treeable. E.g. $GL_n(\mathbb{Z})$, $SL_n(\mathbb{Z})$, $n \geq 3$.
(b) Treeable eq. rel. are not closed under product and ctbl increasing unions. E.g. let $E_{\mathbb{F}_2}^{\mu_1}$ & $E_{\mathbb{Z}}^{\mu_2}$ be orbit eq. rel. induced by free pmp actions of $\mathbb{F}_2$ & $\mathbb{Z}$, resp. Then $E_{\mathbb{F}_2} \times E_{\mathbb{Z}}$ is not $\mu_1 \times \mu_2$-treeable.

Open problem. Is treeability closed under finite-index extensions?



In fact, let $F \subseteq E$ be $[E:F]=2$, i.e. each E-class has exactly 2 F-class. If F is treeable, is E treeable? Maybe μ-treeable?